

On the Stability of a Polling System with an Adaptive Service Mechanism

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Abstract

We consider a single-server cyclic polling system with three queues where the server follows an adaptive rule: if it finds one of queues empty in a given cycle, it decides not to visit that queue in the next cycle. In the case of limited service policies, we prove stability and instability results under some conditions which are sufficient but not necessary, in general. Then we discuss open problems with identifying the exact stability region for models with limited service disciplines: we conjecture that a necessary and sufficient condition for the stability may depend on the whole distributions of the primitive sequences, and illustrate that by examples. We conclude the paper with a section on the stability analysis of a polling system with either gated or exhaustive service disciplines.

Keywords: Polling System, Limited, Gated and Exhaustive Service Disciplines, Stability, Fluid Limits

AMS Classification: 60K25, 68M20

1 Introduction

A standard polling model is a single-server system where the server visits a finite number of queues in cyclic order. The stability and performance analysis of polling models with cyclic and other scheduling policies has been a very popular research topic for several decades. See, e.g. Borst (1995); Boxma et al. (2009); Wierman et al. (2007); Winands et al. (2009) and the lists of references therein for the progress in the studies of polling models. One of the key tools in the modern stability analysis of queueing networks is the fluid limit approach. See, e.g., Chen and Mandelbaum (1991); Rybko and Stolyar (1992); Stolyar (1995); Dai (1995); Dai and Meyn (1995); Down (1996) for the detailed description. This approach involves the use of the functional strong law of large numbers and works perfectly well if one can identify all the limiting parameters/functions there. This holds, in particular, for models where the server's scheduling is state-independent.

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In this paper, we make an attempt to study a model with an adaptive scheduling of the server's visits to queues. Namely, we consider a model with 3 queues and a cyclic policy, but assume in addition that if in some cycle the server finds queue 2 empty, it does not visit this queue in the next cycle. This may make sense if one assumes, in addition, that the direct walking time from queue 1 to queue 3 is smaller than that via queue 2. We consider three types of server's policies/disciplines in queues: limited, gated and exhaustive. We were motivated by the paper Vishnevsky and Semenova (2008) where the authors considered a general model with many queues and proposed a numerical iterative scheme to calculate the mean waiting time. They showed numerically that, in the case of limited service disciplines, the stability region may become bigger by implementing such an adaptive scheduling.

Our original intention was to bring a mathematical accuracy here and to obtain, for the proposed scheme, some necessary and sufficient conditions for the stability which are better than those in non-adaptive schemes. We planned to use the fluid limit approach for the stability. However, we have completed only a part of our programme. First, we derived the evolution differential equations for the fluid limits. Then we got two types of results.

When the service discipline in each queue is either gated or exhaustive, the exact stability region is derived. The stability region in this case is the same as for the standard non-adaptive schemes, and the stability analysis here is a straightforward application of the techniques developed, say, in Dai (1995) and Dai and Meyn (1995). So, a possible advantage of the use of adaptive scheduling is not in the stability, but in the performance: when the polling systems are stable, stationary characteristics under the adaptive schemes may be smaller than those under the non-adaptive schemes.

If the server discipline in each queue is limited, then we can derive tight fluid model equations only in the case where the service discipline in the second queue is 1-limited. For a general limited discipline, we use simple bounds for a fluid model equation to obtain separately sufficient conditions for stability and for instability.

It is known that a sample-path analysis of fluid limits may become inefficient due to several reasons. One of them was discussed, for example, by Foss and Kovalevskii (1999) who considered fluid limits as weak limits of stochastic processes under the linear scaling in time and in space and show, in particular, that these limits may stay random. We will discuss another reason for the limitation of the fluid limit approach which is observed in the polling models: a stability region may depend on the entirely whole distribution of driving sequences. We believe that the gap between the stability and instability conditions is not just a technical limitation of the fluid approximation, but indicates that the conditions obtained are as sharp as one could get based only on the knowledge of the first moments. We guess that, for any set of parameters in the gap, one may propose a system that is stable and another system that is unstable.

The remainder of this paper is organized as follows. In Section 2, we provide a detailed description of the polling system and of the underlying Markov process. Then we present the main results of the paper on stability and instability of the system. In Section 3, we introduce fluid limits for the system with limited service policies, derive dynamical fluid equations and recall known stability and instability criteria via fluid limits. Then we prove in Section 4 the main results, in the case of limited service

policies. Section 5 presents simulation results and a discussion on applicability and accuracy of the fluid model approach for (in)stability of polling systems with adaptive routing and limited service disciplines. In Section 6, we study a fluid model for gated and exhaustive service disciplines and prove the main theorems in this case.

2 Model Description and main results

We consider a polling system with three infinite-buffer queues/stations and a single server. The input stream to queue $k = 1, 2, 3$ is described by interarrival times $\tau_k(n)$ between n th and $(n + 1)$ st customers, $n = 1, 2, \dots$; and n th customer in queue k requires $\sigma_k(n)$ units of time for service. The server visits queues in a cyclic order. We assume that the cycles start from queue 1, and that there are two possible cycle types:

- *standard* cycle: $1 \rightarrow 2 \rightarrow 3 \rightarrow 1$;
- *reduced* cycle: $1 \rightarrow 3 \rightarrow 1$.

A choice of the type of the next cycle on the base of the current cycle information is described in detail later on. In short, the server may decide not to visit queue 2 in the next cycle if it finds that queue empty.

The n th walking (switch-over) time from station k to station $(k + 1)(\bmod 3)$ is denoted by $\xi_k(n)$, while walking times from station 1 to station 3 are $\xi_4(n)$.

A service discipline describes the number of customers that (may be) served during a single visit of the server to a queue. We will assume that the service disciplines in all queues are the same and belong to one of the following three classes:

1. *limited*: upon a visit to queue $k = 1, 2, 3$, the server serves at most $l_k \geq 1$ customers, i.e. it continues working until either a predefined number l_k of customers is served or until the queue becomes empty, whichever occurs first;
2. *gated*: upon a visit to a queue, the server serves all the customer that are present at the moment of the arrival, and only those customers;
3. *exhaustive*: the server continues to serve customers in a queue until the queue becomes empty.

Now we describe inductively the way how does the server choose a type of the next cycle. Here the state of queue 2 (empty or not) within a cycle plays a crucial role.

- The first cycle is of the standard type.
- If cycle n is of the standard type, and
 - if, within this cycle, the server finds queue 2 non-empty and proceeds with the service there, then cycle $n + 1$ is again standard;
 - if queue 2 is empty, the cycle $n + 1$ is reduced.

- If cycle n is reduced, then cycle $n + 1$ is standard.

Stochastic Assumptions: We assume that all sequences of random variables introduced above are mutually independent and that each of these sequences is i.i.d., with a finite first moment. We introduce arrival and service rates at each queue as

$$\lambda_k = \frac{1}{\mathbb{E}\tau_k(1)} > 0, \quad \mu_k = \frac{1}{\mathbb{E}\sigma_k(1)} > 0.$$

Let $\beta_k = \lambda_k/\mu_k$, and $\rho_0 = \sum_{k=1}^3 \beta_k$. Further, we introduce switch-over rates as

$$\nu_k = \frac{1}{\mathbb{E}\xi_k(1)}, \quad k = 1, \dots, 4.$$

Then $\zeta = \sum_{k=1}^3 \nu_k^{-1}$ is the mean switch-over time in the standard cycle, and $\zeta^* = \nu_3^{-1} + \nu_4^{-1}$ is the mean switch-over time in the reduced cycle. We assume that

$$\zeta > \zeta^* \tag{2.1}$$

which makes sense: the server may decide to skip its visit to queue 2 if this may decrease the cycle time.

Representing Markov process: We use

$$X(t) = (Q(t), A(t), B(t), B^0(t), H(t), I(t), C(t))$$

to denote the state of our polling system at time t . The components are described below:

- $Q(t) = (Q_1(t), Q_2(t), Q_3(t))$, where $Q_i(t)$ is the total number of customers that are either waiting in queue i or being served at station i at time t .
- $A(t) = (A_1(t), A_2(t), A_3(t))$, where $A_i(t)$ is the remaining interarrival time of the arrival process to queue i at time t .
- $B(t)$ is the remaining service time of a customer who is being served at time t if the server is in service (and $B(t) = 0$ if the server is walking at time t).
- $B^0(t)$ is the remaining walking time of the server at time t , if the server is walking (and $B^0(t) = 0$ if the server is in service).
- If the server is in service at time t , $H(t)$ is the station the server works at. If the server is walking at time t , $H(t)$ is the station the server is walking to.
- $I(t)$ takes values 0 and 1. It switches from 0 to 1 at the moment when the server arrives at queue 2 and finds it empty, and, vice versa, from 1 to 0 when the server starts its walking from queue 1 to queue 3 in a reduced cycle.

- $C(t)$ is an additional parameter at time t that makes $X(t)$ a Markov process. If the server is in service at time t , then $C(t)$ is the number of service completions at queue $H(t)$ by time t during the current visit to the queue, for limited service disciplines; and the number of arrivals at queue $H(t)$ by time t during the current visit to the queue, for gated service disciplines. In both cases we let $C(t) = 0$ when the server is walking. In the case of exhaustive disciplines, we do not need an extra parameter, so let $C(t) \equiv 0$.

The processes $X = \{X(t) : t \geq 0\}$ are taken to be right-continuous with left limits. It follows from Dai (1995) that X is a strong Markov process whose state space \mathcal{S} is a subset of \mathbb{R}_+^{11} .

The Markov process X is said to be positive Harris recurrent if it possesses a unique stationary distribution. To state the main results of this paper, we make the following assumptions on the interarrival time distributions. For each $k = 1, 2, 3$, we assume that the distribution of random variable $\tau_k(1)$ has an unbounded support, i.e., $\mathbf{P}(\tau_k(1) > t) > 0$, for all $t > 0$. Further, we assume that, for each $k = 1, 2, 3$, the distribution of $\tau_k(1)$ is spread-out, i.e., there exists an integer $n > 0$ and a non-negative function $g(x)$ with $\int_0^\infty g(x) dx > 0$, such that

$$\mathbf{P}\left(a \leq \sum_{i=1}^n \tau_k(i) \leq b\right) \geq \int_a^b g(x) dx, \text{ for any } 0 \leq a < b.$$

We say that a polling system is *stable* if the underlying Markov process is positive Harris recurrent, and *unstable*, otherwise. An unstable polling model is *transient* if $|Q(t)| \rightarrow \infty$ a.s., as $t \rightarrow \infty$.

The following theorems provides sufficient conditions for stability, instability and transience of the polling systems under consideration.

Theorem 1. *A polling system with limited service disciplines is stable if the following three conditions hold:*

$$\rho_0 + \frac{\lambda_2}{l_2} \zeta < 1, \quad (2.2)$$

$$\rho_0 + \min\left(\frac{\lambda_1}{l_1} \frac{\zeta + \zeta^*}{2} + \lambda_2 \frac{\zeta - \zeta^*}{2}, \frac{\lambda_1}{l_1} \zeta\right) < 1, \quad (2.3)$$

$$\rho_0 + \min\left(\frac{\lambda_3}{l_3} \frac{\zeta + \zeta^*}{2} + \lambda_2 \frac{\zeta - \zeta^*}{2}, \frac{\lambda_3}{l_3} \zeta\right) < 1. \quad (2.4)$$

A polling system with either gated or exhaustive service disciplines is stable if $\rho_0 < 1$.

Theorem 2. *A polling system with limited service disciplines is unstable if at least one of the following three inequalities hold, either*

$$\rho_0 + \frac{\lambda_2}{l_2} \zeta \geq 1, \text{ or} \quad (2.5)$$

$$\rho_0 + \frac{\lambda_1}{l_1} \frac{\zeta + \zeta^*}{2} + \frac{\lambda_2}{l_2} \frac{\zeta - \zeta^*}{2} \geq 1, \text{ or} \quad (2.6)$$

$$\rho_0 + \frac{\lambda_3}{l_3} \frac{\zeta + \zeta^*}{2} + \frac{\lambda_2}{l_2} \frac{\zeta - \zeta^*}{2} \geq 1. \quad (2.7)$$

Moreover, the system is transient if one of the inequalities above is strict.

A polling system with either gated or exhaustive service disciplines is unstable if $\rho_0 \geq 1$, and transient if $\rho_0 > 1$.

In two particular cases, the statements of Theorems 1 and 2 do match:

Corollary 1. *A polling system with limited service disciplines and with $l_2 = 1$ is stable if and only if (2.2), (2.3) and (2.4) are satisfied.*

A polling system with gated or exhaustive service disciplines is stable if and only if $\rho_0 < 1$.

3 The fluid model and (in)stability criteria – Limited service disciplines

In this section, we consider a polling system with limited service disciplines. We define fluid limits and derive fluid model equations that are satisfied by fluid limits. Stability and instability criteria are given via the fluid model defined by the fluid model equations. In our construction of the fluid model, we follow the general scheme, see e.g. Chen and Mandelbaum (1991); Rybko and Stolyar (1992); Stolyar (1995); Dai (1995); Dai and Meyn (1995); Down (1996).

First we define processes related to our polling system.

- $T(t) = (T_1(t), T_2(t), T_3(t))$, $t \geq 0$, where $T_k(t)$ is the amount of time the server spends in service at station k during the time interval $[0, t]$.
- $U(t) = (U_1(t), U_2(t), U_3(t), U_4(t))$, $t \geq 0$, where $U_1(t)$ ($U_2(t)$, $U_3(t)$, and $U_4(t)$, respectively) is the amount of time the server spends walking from station 1 to 2 (2 to 3, 3 to 1, and 1 to 3, respectively), during $[0, t]$.

Let $\mathbb{X} = \{(Q(t), T(t), U(t)) : t \geq 0\}$. If the initial state $x \in \mathcal{S}$ of the Markov process X is needed to be displayed explicitly, \mathbb{X}_x is used for the process \mathbb{X} obtained with the initial state x of the Markov process X .

By the strong law of large numbers, for almost all sample paths ω , we have

$$\lim_{n \rightarrow \infty} \frac{1}{n} \sum_{i=1}^n \tau_k(i, \omega) = \frac{1}{\lambda_k}, \quad k = 1, 2, 3, \quad (3.1)$$

$$\lim_{n \rightarrow \infty} \frac{1}{n} \sum_{i=1}^n \sigma_k(i, \omega) = \frac{1}{\mu_k}, \quad k = 1, 2, 3, \quad (3.2)$$

$$\lim_{n \rightarrow \infty} \frac{1}{n} \sum_{i=1}^n \xi_k(i, \omega) = \frac{1}{\nu_k}, \quad k = 1, 2, 3, 4. \quad (3.3)$$

It follows from the same argument as in Dai (1995) that for every sample path ω satisfying (3.1)-(3.3) and every collection $\{x_r : r > 0\}$ of initial states such that $\{|x_r|/r : r > 0\}$ is bounded, there exists a subsequence $r_n \rightarrow \infty$ such that

$\frac{1}{r_n} \mathbb{X}_{x_{r_n}}(r_n \cdot, \omega)$ converges uniformly on any compact subset of $[0, \infty)$ to some limit say $\bar{\mathbb{X}} = (\bar{Q}(\cdot), \bar{T}(\cdot), \bar{U}(\cdot))$. Each such limit $\bar{\mathbb{X}}$ is called a *fluid limit*. In the special case where the sequence of initial states $\{x_r : r > 0\}$ is independent of r , we call the limit a *fluid limit with fixed initial state*. Both types of fluid limits are used in our subsequent stability analysis.

It is known (see, e.g., Bramson (1998)) that, in the analysis of stability via fluid limits, it is sufficient to consider the so-called undelayed fluid limits only, i.e. limits that satisfy the assumption

$$\lim_{r \rightarrow \infty} \frac{1}{r} (|a_r| + b_r + b_r^0) = 0, \quad (3.4)$$

where a_r , b_r and b_r^0 are subvectors of the initial state $x_r = (q_r, a_r, b_r, b_r^0, h_r, i_r, c_r)$. From now on, we consider only undelayed fluid limits.

If $\bar{\mathbb{X}}$ is a fluid limit obtained from a sequence of initial states $\{x_r\}$ satisfying (3.4), then all of $\bar{Q}_k(\cdot)$, $k = 1, 2, 3$, $\bar{T}_k(\cdot)$, $k = 1, 2, 3$, and $\bar{U}_k(\cdot)$, $k = 1, 2, 3, 4$, are Lipschitz continuous functions. Hence they are absolutely continuous and thus differentiable almost everywhere with respect to the Lebesgue measure. We say that t is a *regular point* of $\bar{\mathbb{X}}$ if all components of $\bar{\mathbb{X}}$ are differentiable at t . In the rest of the paper, we implicitly assume that t is a regular point whenever the derivative of a component of $\bar{\mathbb{X}}$ is involved.

The following theorem presents equations that are satisfied by the fluid limits.

Lemma 1. *For every fluid limit $\bar{\mathbb{X}}(t) = (\bar{Q}(t), \bar{T}(t), \bar{U}(t))$, the following equations hold:*

$$\bar{Q}_k(t) = \bar{Q}_k(0) + \lambda_k t - \mu_k \bar{T}_k(t), \quad k = 1, 2, 3, \quad t \geq 0; \quad (3.5)$$

$$\bar{Q}_k(t) \geq 0, \quad k = 1, 2, 3, \quad t \geq 0; \quad (3.6)$$

$$\bar{T}_k(\cdot) \text{ and } \bar{U}_j(\cdot) \text{ are nondecreasing, } \quad k = 1, 2, 3, \quad j = 1, 2, 3, 4; \quad (3.7)$$

$$\sum_{k=1}^3 \bar{T}_k(t) + \sum_{j=1}^4 \bar{U}_j(t) = t, \quad t \geq 0; \quad (3.8)$$

$$\nu_1 \bar{U}_1(t) = \nu_2 \bar{U}_2(t), \quad t \geq 0; \quad (3.9)$$

$$\nu_3 \bar{U}_3(t) = \nu_1 \bar{U}_1(t) + \nu_4 \bar{U}_4(t), \quad t \geq 0; \quad (3.10)$$

$$\nu_1 \bar{U}'_1(t) + \nu_4 \bar{U}'_4(t) \geq \frac{\mu_1}{l_1} \bar{T}'_1(t), \quad t \geq 0; \quad (3.11)$$

$$\nu_2 \bar{U}'_2(t) \geq \frac{\mu_2}{l_2} \bar{T}'_2(t), \quad t \geq 0; \quad (3.12)$$

$$\nu_3 \bar{U}'_3(t) \geq \frac{\mu_3}{l_3} \bar{T}'_3(t), \quad t \geq 0; \quad (3.13)$$

$$\frac{\mu_2}{l_2} \bar{T}'_2(t) \leq \nu_1 \bar{U}'_1(t) - \nu_4 \bar{U}'_4(t) \leq \mu_2 \bar{T}'_2(t), \quad t \geq 0; \quad (3.14)$$

$$\text{If } \bar{Q}_1(t) > 0, \text{ then } \nu_1 \bar{U}'_1(t) + \nu_4 \bar{U}'_4(t) = \frac{\mu_1}{l_1} \bar{T}'_1(t), \quad t \geq 0; \quad (3.15)$$

$$\text{If } \bar{Q}_2(t) > 0, \text{ then } \bar{U}'_4(t) = 0 \text{ and } \nu_2 \bar{U}'_2(t) = \frac{\mu_2}{l_2} \bar{T}'_2(t), \quad t \geq 0; \quad (3.16)$$

$$\text{If } \bar{Q}_3(t) > 0, \text{ then } \nu_3 \bar{U}'_3(t) = \frac{\mu_3}{l_3} \bar{T}'_3(t), \quad t \geq 0. \quad (3.17)$$

Proof. All equations in the theorem are obtained through the standard procedure. We provide below proofs only for equations (3.10), (3.11), (3.14) and (3.15), and omit all other proofs. Let

- $D(t) = (D_1(t), D_2(t), D_3(t))$, where $D_i(t)$ is the number of service completions at station i by time t ;
- $M(t) = (M_1(t), M_2(t), M_3(t))$, where $M_i(t)$ is the number of service completions at station i if the server spends t units of time working at station i ;
- $E(t) = (E_1(t), E_2(t), E_3(t), E_4(t))$, where $E_1(t)$ ($E_2(t)$, $E_3(t)$, and $E_4(t)$, respectively) is the number of switch-over completions from station 1 to station 2 (from 2 to 3, from 3 to 1, and from 1 to 3, respectively) by time t ;
- $N(t) = (N_1(t), N_2(t), N_3(t), N_4(t))$, where $N_1(t)$ ($N_2(t)$, $N_3(t)$, and $N_4(t)$, respectively) is the number of switch-over completions from station 1 to station 2 (from 2 to 3, from 3 to 1, and from 1 to 3, respectively) if the server spends t units of time walking from station 1 to station 2 (from 2 to 3, from 3 to 1, and from 1 to 3, respectively).

Then

$$D_i(t) = M(T_i(t)), \quad t \geq 0, \quad i = 1, 2, 3, \quad (3.18)$$

$$E_i(t) = N(U_i(t)), \quad t \geq 0, \quad i = 1, 2, 3, 4, \quad (3.19)$$

and (3.2) and (3.3) imply

$$\lim_{t \rightarrow \infty} \frac{M_i(t)}{t} = \mu_i, \quad i = 1, 2, 3, \quad (3.20)$$

$$\lim_{t \rightarrow \infty} \frac{N_i(t)}{t} = \nu_i, \quad i = 1, 2, 3, 4. \quad (3.21)$$

Now we prove (3.10), (3.11), (3.14) and (3.15).

- (3.10): Since $|E_1(t) + E_4(t) - E_3(t)| \leq 1$, (3.19) yields

$$|N_1(U_1(t)) + N_4(U_4(t)) - N_3(U_3(t))| \leq 1.$$

Applying fluid limits to the above equation, we obtain (3.10) with the help of (3.21).

- (3.11): For $0 \leq t_1 \leq t_2$, we have

$$E_1(t_2) - E_1(t_1) + E_4(t_2) - E_4(t_1) + 1 \geq \frac{D_1(t_2) - D_1(t_1)}{l_1}.$$

Substituting (3.18) and (3.19) into the above equation yields

$$N_1(U_1(t_2)) - N_1(U_1(t_1)) + N_4(U_4(t_2)) - N_4(U_4(t_1)) + 1 \geq \frac{M_1(T_1(t_2)) - M_1(T_1(t_1))}{l_1}.$$

Applying fluid limits to the above equation, we obtain (3.11) with the help of (3.20) and (3.21).

- (3.14): Recall that if no customer is served at station 2 during a standard cycle, then the next cycle is reduced. Conversely, if at least one customer is served at station 2 during a standard cycle, then the next cycle is standard. Therefore, for $0 \leq t_1 \leq t_2$, the number of cycles with at least one service completion at station 2 during (t_1, t_2) differs at most by 1 from $(E_1(t_2) - E_1(t_1)) - (E_4(t_2) - E_4(t_1))$. Therefore

$$\begin{aligned}
& (E_1(t_2) - E_1(t_1)) - (E_4(t_2) - E_4(t_1)) - 1 \\
& \leq D_2(t_2) - D_1(t_1) \\
& \leq l_2((E_1(t_2) - E_1(t_1)) - (E_4(t_2) - E_4(t_1)) + 1).
\end{aligned}$$

Substituting (3.18) and (3.19) into the above equation and applying fluid limits yields (3.14).

- (3.15): Let $0 \leq t_1 \leq t_2$. If $Q_1(t) > 0$ for all $t \in (t_1, t_2)$, then

$$\left| E_1(t_2) - E_1(t_1) + E_4(t_2) - E_4(t_1) - \frac{D_1(t_2) - D_1(t_1)}{l_1} \right| \leq 1.$$

Hence if $Q_1(t) > 0$ for all $t \in (t_1, t_2)$, then

$$\left| N_1(U_1(t_2)) - N_1(U_1(t_1)) + N_4(U_4(t_2)) - N_4(U_4(t_1)) - \frac{M_1(T_1(t_2)) - M_1(T_1(t_1))}{l_1} \right| \leq 1.$$

Applying fluid limits to the above equation, we obtain (3.15). □

We call the equations (3.5)-(3.17) the *fluid model equations* and call a solution $\bar{\mathbb{X}} = \{(\bar{Q}(t), \bar{T}(t), \bar{U}(t)) : t \geq 0\}$, of the fluid model equations a *fluid model solution*. Note that any fluid limit with fixed initial state necessarily has $\bar{Q}(0) = 0$. Thus these fluid limits form a subset of fluid model solutions with $\bar{Q}(0) = 0$. The following definitions and lemmas indicate the usefulness of different types of fluid limits.

Definition 1. (i) *The fluid model is stable if there exists a $\delta > 0$ such that for each fluid model solution $\bar{\mathbb{X}}$ with $|\bar{Q}(0)| \leq 1$, $\bar{Q}(t) = 0$ for $t \geq \delta$.*

(ii) *The fluid model is weakly unstable if there exists a $\delta > 0$ such that, for each fluid model solution $\bar{\mathbb{X}}$ with $\bar{Q}(0) = 0$, $\bar{Q}(\delta) \neq 0$.*

The reasoning used in Dai (1995,1996), can be applied easily to our polling system to obtain the following criteria.

Lemma 2. (Dai (1995)) *If the fluid model is stable, then the stochastic polling system is stable too.*

Lemma 3. (Dai (1996)) *If the fluid model is weakly unstable, then the stochastic polling system is transient.*

We present now a weaker instability criterion than Lemma 3, by applying to our polling systems the arguments first introduced in Dai et al. (2007), see Lemma 5 below. If we assume *a priori* that the process X is positive Harris recurrent, then any fluid limit with fixed initial state must obey an extra dynamical equation, which augments the fluid model equations presented in (3.5)-(3.17). Let $F_i(t)$ be the number of server's visits to station i when it is empty, in the time interval $(0, t)$. By the theory of Markov regenerative processes, if X is positive Harris recurrent, then there are positive numbers f_i , $i = 1, 2, 3$, such that

$$\lim_{t \rightarrow \infty} \frac{F_i(t)}{t} = f_i, \quad i = 1, 2, 3, \quad (3.22)$$

with probability 1.

Lemma 4. *Suppose that the Markov process X is positive Harris recurrent, and a fluid limit with fixed initial state $\bar{X}(t) = (\bar{Q}(t), \bar{T}(t), \bar{U}(t))$ is driven from a sample path ω that satisfies (3.22). Then the following inequalities hold: For $t \geq 0$,*

$$\nu_1 \bar{U}'_1(t) + \nu_4 \bar{U}'_4(t) > \frac{\mu_1}{l_1} \bar{T}'_1(t); \quad (3.23)$$

$$\nu_2 \bar{U}'_2(t) > \frac{\mu_2}{l_2} \bar{T}'_2(t); \quad (3.24)$$

$$\nu_3 \bar{U}'_3(t) > \frac{\mu_3}{l_3} \bar{T}'_3(t). \quad (3.25)$$

Proof. We have

$$\begin{aligned} D_1(t_2) - D_1(t_1) &\leq l_1((E_1(t_2) - E_1(t_1)) + (E_4(t_2) - E_4(t_1)) - (F_1(t_2) - F_1(t_1)) + 1), \\ D_2(t_2) - D_2(t_1) &\leq l_2((E_2(t_2) - E_2(t_1)) - (F_2(t_2) - F_2(t_1)) + 1), \\ D_3(t_2) - D_3(t_1) &\leq l_3((E_3(t_2) - E_3(t_1)) - (F_3(t_2) - F_3(t_1)) + 1). \end{aligned}$$

Substituting (3.18) and (3.19) into the above equations and applying fluid limits leads to

$$\begin{aligned} \mu_1 \bar{T}'_1(t) &\leq l_1(\nu_1 \bar{U}'_1(t) + \nu_4 \bar{U}'_4(t) - f_1), \\ \mu_2 \bar{T}'_2(t) &\leq l_2(\nu_2 \bar{U}'_2(t) - f_2), \\ \mu_3 \bar{T}'_3(t) &\leq l_3(\nu_3 \bar{U}'_3(t) - f_3). \end{aligned}$$

Since $f_i > 0$, $i = 1, 2, 3$, Lemma 4 is proved. \square

We call the union of two systems of equations and inequalities (3.5)-(3.17) and (3.23)-(3.25) the *augmented fluid model equations* and call a solution \bar{X} , to these union an *augmented fluid model solution*.

Definition 2. *The augmented fluid model is weakly unstable if there exists a $\delta > 0$ such that for each augmented fluid model solution \bar{X} , with $\bar{Q}(0) = 0$, $\bar{Q}(\delta) \neq 0$.*

Suppose that the augmented fluid model is weakly unstable but the Markov process X is positive Harris recurrent. Since the augmented fluid model equations are satisfied

by every fluid limit which is a limit of scaled sample paths with fixed initial state, the argument in Dai (1996) implies that the process is transient in the sense that, $|Q(t)| \rightarrow \infty$ as $t \rightarrow \infty$ with probability 1, which is a contradiction. Therefore we obtain the following instability criterion.

Lemma 5. (*Dai et al. (2007)*) *If the augmented fluid model is weakly unstable, then the stochastic system is unstable.*

4 Proof of Theorems 1 and 2 for limited service disciplines

For a polling system with limited service disciplines, Theorems 1 and 2 follow from Propositions 1, 2 and 3 below and the (in)stability criteria, Lemmas 2, 3 and 5.

Proposition 1. *For the polling system with limited service disciplines, the fluid model is stable if (2.2), (2.3) and (2.4) are satisfied.*

Proposition 2. *For the polling system with limited service disciplines, the augmented fluid model is weakly unstable if (2.5), (2.6) or (2.7) holds.*

Proposition 3. *For the polling system with limited service disciplines, the fluid model is weakly unstable if at least one of the inequalities (2.5), (2.6) or (2.7) is strict.*

The remainder of this section is devoted to the proof of Propositions 1, 2 and 3. For a fluid model solution $\bar{\mathbb{X}}$, let

$$J(t) \equiv \{k \in \{1, 2, 3\} : \bar{Q}_k(t) > 0\}, \quad t \geq 0. \quad (4.1)$$

Lemma 6. *For each fluid model solution $\bar{\mathbb{X}}$, if $\bar{Q}_2(t) > 0$, then*

$$\bar{T}'_j(t) = \frac{l_j}{\mu_j} \frac{1 - \rho_0 + \sum_{k \in J(t)} \beta_k}{\zeta + \sum_{k \in J(t)} \frac{l_k}{\mu_k}}, \quad j \in J(t). \quad (4.2)$$

Proof. Let $\bar{\mathbb{X}}$ be a fluid model solution. Suppose that $\bar{Q}_2(t) > 0$. If $\bar{Q}_j(t) > 0$, then, according to (3.5), (3.9), (3.10) and (3.15)–(3.17), we have

$$\begin{aligned} \bar{U}'_k(t) &= \frac{\mu_j}{l_j} \frac{1}{\nu_k} \bar{T}'_j(t), \quad k = 1, 2, 3, \\ \bar{T}'_k(t) &= \begin{cases} \frac{l_k}{\mu_k} \frac{\mu_j}{l_j} \bar{T}'_j(t), & k \in J(t), \\ \beta_k, & k \in \{1, 2, 3\} \setminus J(t). \end{cases} \end{aligned}$$

Substituting the above equations into (3.8) yields (4.2). \square

Lemma 7. *For each fluid model solution $\bar{\mathbb{X}}$, if $\bar{Q}_2(t) = 0$, then*

$$\bar{T}'_j(t) \geq \frac{l_j}{\mu_j} \frac{1 - \rho_0 + \sum_{k \in J(t)} \beta_k - \lambda_2 \frac{\zeta - \zeta^*}{2}}{\frac{\zeta + \zeta^*}{2} + \sum_{k \in J(t)} \frac{l_k}{\mu_k}}, \quad j \in J(t); \quad (4.3)$$

$$\bar{T}'_j(t) \geq \frac{l_j}{\mu_j} \frac{1 - \rho_0 + \sum_{k \in J(t)} \beta_k}{\zeta + \sum_{k \in J(t)} \frac{l_k}{\mu_k}}, \quad j \in J(t). \quad (4.4)$$

Proof. Let $\bar{\mathbb{X}}$ be a fluid model solution. Suppose that $\bar{Q}_2(t) = 0$. According to (3.5), (3.9), (3.10) and (3.15)–(3.17), we have

$$\bar{T}'_k(t) = \frac{l_k}{\mu_k} \frac{\mu_j}{l_j} \bar{T}'_j(t), \quad \text{if } \bar{Q}_j(t) > 0 \text{ and } k \in J(t), \quad (4.5)$$

$$\bar{T}'_k(t) = \beta_k, \quad \text{if } k \in \{1, 2, 3\} \setminus J(t). \quad (4.6)$$

Clearly, for any fixed k , the right-hand side in (4.5) is the same for all j with $\bar{Q}_j(t) > 0$.

According to (3.9)–(3.13) and (3.15)–(3.17), we have

$$\sum_{k=1}^4 \bar{U}'_k(t) = \frac{\mu_j}{l_j} \bar{T}'_j(t) \zeta - \nu_4 \bar{U}'_4(t) (\zeta - \zeta^*), \quad \text{if } \bar{Q}_j(t) > 0. \quad (4.7)$$

By (3.7), (3.10)–(3.15) and (4.7),

$$\sum_{k=1}^4 \bar{U}'_k(t) \leq \frac{\mu_j}{l_j} \bar{T}'_j(t) \frac{\zeta + \zeta^*}{2} + \lambda_2 \frac{\zeta - \zeta^*}{2}, \quad \text{if } \bar{Q}_j(t) > 0, \quad (4.8)$$

$$\sum_{k=1}^4 \bar{U}'_k(t) \leq \frac{\mu_j}{l_j} \bar{T}'_j(t) \zeta, \quad \text{if } \bar{Q}_j(t) > 0. \quad (4.9)$$

Substituting (4.5), (4.6) and (4.8) into (3.8) yields (4.3). Substituting (4.5), (4.6) and (4.9) into (3.8) yields (4.4). \square

Lemma 8. *For each fluid model solution $\bar{\mathbb{X}}$,*

$$\bar{T}'_2(t) \leq \frac{l_2}{\mu_2} \frac{1 - \rho_0 + \sum_{k \in J(t) \cup \{2\}} \beta_k}{\zeta + \sum_{k \in J(t) \cup \{2\}} \frac{l_k}{\mu_k}}; \quad (4.10)$$

$$\bar{T}'_j(t) \leq \frac{l_j}{\mu_j} \frac{1 - \rho_0 + \sum_{k \in J(t) \cup \{j\}} \beta_k - \frac{\lambda_2}{l_2} \frac{\zeta - \zeta^*}{2}}{\frac{\zeta + \zeta^*}{2} + \sum_{k \in J(t) \cup \{j\}} \frac{l_k}{\mu_k}}, \quad j = 1, 3. \quad (4.11)$$

Proof. Let $\bar{\mathbb{X}}$ be a fluid model solution. According to (3.5), (3.9), (3.10) and (3.15)–(3.17), we have

$$\bar{T}'_k(t) \geq \frac{l_k}{\mu_k} \frac{\mu_j}{l_j} \bar{T}'_j(t), \quad k \in J(t), \quad j = 1, 2, 3, \quad (4.12)$$

$$\bar{T}'_k(t) = \beta_k, \quad k \in \{1, 2, 3\} \setminus J(t). \quad (4.13)$$

According to (3.9)–(3.13) and (3.15)–(3.17), we have

$$\sum_{k=1}^4 \bar{U}'_k(t) \geq \nu_3 \bar{U}'_3(t) \zeta - \nu_4 \bar{U}'_4(t) (\zeta - \zeta^*), \quad j = 1, 2, 3, \quad (4.14)$$

$$\sum_{k=1}^4 \bar{U}'_k(t) \geq \frac{\mu_2}{l_2} \bar{T}'_2(t) \zeta. \quad (4.15)$$

By (3.7), (3.10)-(3.15), (3.17) and (4.14),

$$\sum_{k=1}^4 \bar{U}'_k(t) \geq \frac{\mu_j}{l_j} \bar{T}'_j(t) \frac{\zeta + \zeta^*}{2} + \frac{\lambda_2}{l_2} \frac{\zeta - \zeta^*}{2}. \quad (4.16)$$

Substituting (4.12), (4.13) and (4.15) into (3.8) yields (4.10). Substituting (4.12), (4.13) and (4.16) into (3.8) yields (4.11). \square

Lemma 9. *For each augmented fluid model solution $\bar{\mathbb{X}}$,*

$$\bar{T}'_2(t) < \frac{l_2}{\mu_2} \frac{1 - \rho_0 + \beta_2}{\zeta + \frac{l_2}{\mu_2}}; \quad (4.17)$$

$$\bar{T}'_j(t) < \frac{l_j}{\mu_j} \frac{1 - \rho_0 + \beta_j - \frac{\lambda_2}{l_2} \frac{\zeta - \zeta^*}{2}}{\frac{\zeta + \zeta^*}{2} + \frac{l_j}{\mu_j}}, \quad j = 1, 3. \quad (4.18)$$

Proof. Let $\bar{\mathbb{X}}$ be an augmented fluid model solution. By (3.15)-(3.17) and (3.23)-(3.25), we have $\bar{Q}_k(t) = 0$, $k = 1, 2, 3$, and, by (3.5),

$$\bar{T}'_k(t) = \beta_k, \quad t \geq 0, \quad k = 1, 2, 3. \quad (4.19)$$

By (3.9) and (3.10) and (3.23)-(3.25),

$$\sum_{k=1}^4 \bar{U}'_k(t) > \frac{\mu_2}{l_2} \bar{T}'_2(t) \zeta, \quad (4.20)$$

$$\sum_{k=1}^4 \bar{U}'_k(t) > \frac{\mu_j}{l_j} \bar{T}'_j(t) \zeta - \nu_4 \bar{U}'_4(t) (\zeta - \zeta^*), \quad j = 1, 3. \quad (4.21)$$

Substituting (4.19) and (4.20) into (3.8) leads to (4.17). Substituting (4.19) and (4.21) into (3.8) leads to (4.18). \square

Now we are ready to prove Propositions 1, 2 and 3.

Proof of Proposition 1. Suppose that (2.2), (2.3) and (2.4) are satisfied. For a fluid model solution $\bar{\mathbb{X}}$, let

$$W(t) \equiv \sum_{k=1}^3 \frac{\bar{Q}_k(t)}{\mu_k}.$$

Then $W(t) = 0$ if and only if $|\bar{Q}(t)| = 0$. We prove that the fluid model is stable by showing that there is a $\delta > 0$ such that for each fluid model solution $\bar{\mathbb{X}}$, with $|\bar{Q}(0)| \leq 1$, $W(t) = 0$ for $t \geq \delta$. The proof proceeds through 3 steps.

Step 1. Let m be the number of indices j in $\{1, 2, 3\}$ such that $\frac{\lambda_j}{l_j} \leq \frac{\lambda_2}{l_2}$, and let j_1, \dots, j_m be the indices enumerated so that $\frac{\lambda_{j_1}}{l_{j_1}} \leq \frac{\lambda_{j_2}}{l_{j_2}} \leq \dots \leq \frac{\lambda_{j_m}}{l_{j_m}}$. Then there exist $\delta_k \geq 0$, $1 \leq k \leq m$, such that for each fluid model solution $\bar{\mathbb{X}}$ with $|\bar{Q}(0)| \leq 1$,

$$\bar{Q}_j(t) = 0 \text{ for } t \geq \delta_k \text{ and } j \in \{j_1, \dots, j_k\}. \quad (4.22)$$

Proof. Let $\delta_0 = 0$. For $k = 0$, (4.22) holds trivially for each fluid model solution $\bar{\mathbb{X}}$ with $|\bar{Q}(0)| \leq 1$. For $1 \leq k \leq m$, suppose that there exists $\delta_{k-1} \geq 0$ such that, for each fluid model solution $\bar{\mathbb{X}}$ with $|\bar{Q}(0)| \leq 1$, $\bar{Q}_j(t) = 0$ for $t \geq \delta_{k-1}$ and $j \in \{j_1, \dots, j_{k-1}\}$. Suppose that $\bar{\mathbb{X}}$ is a fluid model solution with $|\bar{Q}(0)| \leq 1$. According to (4.2) and (4.4), if $\bar{Q}_{j_k}(t) > 0$, then

$$\begin{aligned}\bar{Q}'_{j_k}(t) &\leq \lambda_{j_k} - l_{j_k} \frac{1 - \rho_0 + \sum_{i \in J(t)} \beta_i}{\zeta + \sum_{i \in J(t)} \frac{l_i}{\mu_i}} \\ &= \frac{\lambda_{j_k} \zeta + \lambda_{j_k} \sum_{i \in J(t)} \frac{l_i}{\mu_i} - l_{j_k} (1 - \rho_0 + \sum_{i \in J(t)} \beta_i)}{\zeta + \sum_{i \in J(t)} \frac{l_i}{\mu_i}}.\end{aligned}\quad (4.23)$$

Since $\{j_1, \dots, j_{k-1}\} \cap J(t) = \emptyset$ for $t \geq \delta_{k-1}$, (4.23) leads to

$$Q'_{j_k}(t) \leq -\frac{l_{j_k} (1 - \rho_0 - \frac{\lambda_{j_k} \zeta}{l_{j_k}})}{\zeta + \sum_{i \in J(t)} \frac{l_i}{\mu_i}}, \quad \text{if } t \geq \delta_{k-1} \text{ and } \bar{Q}_{j_k}(t) > 0.$$

Hence

$$\bar{Q}'_{j_k}(t) \leq -\epsilon_k, \quad \text{if } t \geq \delta_{k-1} \text{ and } \bar{Q}_{j_k}(t) > 0,$$

where

$$\epsilon_k \equiv \frac{l_{j_k} (1 - \rho_0 - \frac{\lambda_{j_k} \zeta}{l_{j_k}})}{\zeta + \sum_{i=1}^3 \frac{l_i}{\mu_i}} > 0.$$

According to (3.5) and (3.7), we have

$$Q_{j_k}(\delta_{k-1}) \leq 1 + \lambda_{j_k} \delta_{k-1}.$$

Hence $\bar{Q}_{j_k}(t) = 0$ for $t \geq \delta_k$, where $\delta_k \equiv \delta_{k-1} + \frac{1 + \lambda_{j_k} \delta_{k-1}}{\epsilon_k}$. The proof is completed by induction on k . \square

Step 2. There exists an $\epsilon > 0$ such that, for each fluid model solution $\bar{\mathbb{X}}$, $W'(t) \leq -\epsilon$ if $W(t) > 0$ and $\bar{Q}_2(t) = 0$.

Proof. It is easily proved that at least one of the following inequalities holds:

$$\rho_0 + \frac{\lambda_j}{l_j} \zeta < 1, \quad j = 1, 3, \quad (4.24)$$

$$\rho_0 + \frac{\lambda_j \zeta + \zeta^*}{l_j} + \lambda_2 \frac{\zeta - \zeta^*}{2} < 1, \quad j = 1, 3. \quad (4.25)$$

Let $\bar{\mathbb{X}}$ be a fluid model solution. Suppose that $W(t) > 0$ and $\bar{Q}_2(t) = 0$. By (3.5),

$$W'(t) = \sum_{k \in J(t)} \beta_k - \sum_{k \in J(t)} \bar{T}'_k(t). \quad (4.26)$$

First suppose that (4.24) holds. Substituting (4.4) into (4.26) leads to

$$W'(t) \leq \frac{\sum_{k \in J(t)} \frac{l_k}{\mu_k} (\rho_0 + \frac{\lambda_k}{l_k} \zeta - 1)}{\zeta + \sum_{k \in J(t)} \frac{l_k}{\mu_k}}.$$

Hence $W'(t) \leq -\epsilon$, where

$$\epsilon \equiv \min_{K \in \{\{1\}, \{3\}, \{1,3\}\}} \frac{\sum_{k \in K} \frac{l_k}{\mu_k} (1 - \rho_0 - \frac{\lambda_k}{l_k} \zeta)}{\zeta + \sum_{k \in K} \frac{l_k}{\mu_k}} > 0.$$

Next suppose that (4.25) holds. Substituting (4.3) into (4.26) leads to

$$W'(t) \leq \frac{\sum_{k \in J(t)} \frac{l_k}{\mu_k} (\rho_0 + \frac{\lambda_k}{l_k} \frac{\zeta + \zeta^*}{2} + \lambda_2 \frac{\zeta - \zeta^*}{2} - 1)}{\frac{\zeta + \zeta^*}{2} + \sum_{k \in J(t)} \frac{l_k}{\mu_k}}.$$

Hence $W'(t) \leq -\epsilon$, where

$$\epsilon \equiv \min_{K \in \{\{1\}, \{3\}, \{1,3\}\}} \frac{\sum_{k \in K} \frac{l_k}{\mu_k} (1 - \rho_0 - \frac{\lambda_k}{l_k} \frac{\zeta + \zeta^*}{2} - \lambda_2 \frac{\zeta - \zeta^*}{2})}{\frac{\zeta + \zeta^*}{2} + \sum_{k \in K} \frac{l_k}{\mu_k}} > 0.$$

□

Step 3. There is a $\delta > 0$ such that, for each fluid model solution $\bar{\mathbb{X}}$ with $|\bar{Q}(0)| \leq 1$, $W(t) = 0$ for $t \geq \delta$.

Proof. Suppose that $\bar{\mathbb{X}}$ is a fluid model solution with $|\bar{Q}(0)| \leq 1$. Then $W(0) \leq \max_{1 \leq k \leq 3} \frac{1}{\mu_k}$. According to (3.5) and (3.7), we have

$$W(\delta_m) \leq \max_{1 \leq k \leq 3} \frac{1}{\mu_k} + \rho_0 \delta_m.$$

By Steps 1 and 2,

$$W'(t) \leq -\epsilon, \quad \text{if } t \geq \delta_m \text{ and } W(t) > 0.$$

Hence $W(t) = 0$ for $t \geq \delta$, where $\delta \equiv \delta_m + \frac{1}{\epsilon} (\max_{1 \leq k \leq 3} \frac{1}{\mu_k} + \rho_0 \delta_m)$. □

Proof of Proposition 2. Suppose that at least one of (2.5), (2.6) or (2.7) holds. Let $\bar{\mathbb{X}}$ be an augmented fluid model solution with $\bar{Q}(0) = 0$. By (3.5), (4.17) and (4.18), we have

$$\begin{aligned} \bar{Q}'_2(t) &> \lambda_2 - l_2 \frac{1 - \rho_0 + \beta_2}{\zeta + \frac{l_2}{\mu_2}} \\ &= \frac{l_2 (\frac{\lambda_2}{l_2} \zeta - 1 + \rho_0)}{\zeta + \frac{l_2}{\mu_2}}, \\ \bar{Q}'_j(t) &> \lambda_j - l_j \frac{1 - \rho_0 + \beta_j - \frac{\lambda_2}{l_2} \frac{\zeta - \zeta^*}{2}}{\frac{\zeta + \zeta^*}{2} + \frac{l_j}{\mu_j}} \\ &= \frac{l_j (\frac{\lambda_j}{l_j} \frac{\zeta + \zeta^*}{2} + \frac{\lambda_2}{l_2} \frac{\zeta - \zeta^*}{2} - 1 + \rho_0)}{\frac{\zeta + \zeta^*}{2} + \frac{l_j}{\mu_j}}, \quad j = 1, 3. \end{aligned}$$

Hence there exists $j \in \{1, 2, 3\}$ such that $\bar{Q}'_j(t) > 0$ for all $t \geq 0$, which implies that the augmented fluid model is weakly unstable. \square

Proof of Proposition 3. Suppose that at least one of the inequalities (2.5), (2.6) or (2.7) is strict. Let $\bar{\mathbb{X}}$ be a fluid model solution with $\bar{Q}(0) = 0$.

First suppose that $\frac{\lambda_2}{l_2} > \frac{\lambda_k}{l_k}$, $k = 1, 3$. By (3.5) and (4.10), we have

$$\begin{aligned}\bar{Q}'_2(t) &\geq \lambda_2 - l_2 \frac{1 - \rho_0 + \sum_{k \in J(t) \cup \{2\}} \beta_k}{\zeta + \sum_{k \in J(t) \cup \{2\}} \frac{l_k}{\mu_k}} \\ &\geq \frac{l_2(\frac{\lambda_2}{l_2}\zeta - 1 + \rho_0)}{\zeta + \sum_{k \in J(t) \cup \{2\}} \frac{l_k}{\mu_k}} \\ &> 0,\end{aligned}$$

which implies that the fluid model is weakly unstable.

Next suppose that there exists $j \in \{1, 3\}$ such that $\frac{\lambda_j}{l_j} \geq \frac{\lambda_k}{l_k}$, $k = 1, 2, 3$. By (3.5) and (4.11), we have

$$\begin{aligned}\bar{Q}'_j(t) &\geq \lambda_j - l_j \frac{1 - \rho_0 + \sum_{k \in J(t) \cup \{j\}} \beta_k - \frac{\lambda_2}{l_2} \frac{\zeta - \zeta^*}{2}}{\frac{\zeta + \zeta^*}{2} + \sum_{k \in J(t) \cup \{j\}} \frac{l_k}{\mu_k}} \\ &\geq \frac{l_j(\frac{\lambda_j}{l_j} \frac{\zeta + \zeta^*}{2} + \frac{\lambda_2}{l_2} \frac{\zeta - \zeta^*}{2} - 1 + \rho_0)}{\frac{\zeta + \zeta^*}{2} + \sum_{k \in J(t) \cup \{j\}} \frac{l_k}{\mu_k}} \\ &> 0,\end{aligned}$$

which implies that the fluid model is weakly unstable. \square

5 Discussion on the fluid model for limited service disciplines and simulation results

The upper and lower bounds for the fluid model equation (3.14) is not tight if $l_2 > 1$. From the fluid model equations (3.5)-(3.17), it can be observed that for each fluid model solution $\bar{\mathbb{X}}$, $\bar{Q}'(t)$ is determined by $\bar{U}'_4(t)$, $J(t)$, and the model parameters λ_j , μ_j , l_j , $j = 1, 2, 3$, and ν_k , $k = 1, 2, 3, 4$.

For the case $l_2 > 1$, we conjecture that $\bar{U}'_4(t)$ is not determined by only $J(t)$ and the model parameters but may also depend on the distributions of the driving sequences, the inter-arrival, the service and the switch-over times.

To justify our conjecture, we present here two sets of simulation results for the fluid limits in the system with limited service disciplines. Specifically we observe fluid limits with $\bar{Q}_2(t) = 0$, $\bar{Q}_1(t) > 0$ and $\bar{Q}_3(t) = 0$.

We remind that the condition $\bar{Q}_1(t) > 0$ in a fluid limit corresponds to the condition in the real system that the first queue is infinitely large and that during each visit to this queue the server serves exactly l_1 customers. The goal is to find, in the long run, the fraction of time, u_4 , the server is in the switch-over regime from queue 1 to queue 3, and the fraction of the number of the reduced cycles, p , among all.

5.1 First set of examples

We present five simulation Examples where we vary only one of the distributions of the system (i.e. that of the inter-arrival times to queue 2).

In the first three Examples, we are keeping the first moment fixed and show that both fractions may differ significantly. These examples illustrate that the stability conditions in the system under consideration are not determined, in general, by the first moments of the distributions of the primitive sequences.

We go further and show by example that the knowledge of the first two moments is also not enough. We present Example 4 where the distribution of inter-arrival times to queue 2 have the same 1st and 2nd moments with exponential distribution from Example 3, but here also the fractions of interest significantly differ.

Finally, we finish with showing that even the knowledge of the first three moments is insufficient. In Example 5, the distribution of inter-arrival times to queue 2 have the same the first, the second and the third moments with exponential distribution from Example 3, but again with different fractions of interest.

So, our conjecture is that the fractions of interest may depend on the entirely whole distributions of the driving sequences, and the knowledge of any finite number of moments is not sufficient to determine the stability region precisely.

We consider the following system parameters:

$$\lambda_2 = \lambda_3 = \frac{1}{4}, \nu_1 = 2, \nu_3 = 1, \nu_4 = 3, \nu_2 = +\infty, \mu_2 = 1, \mu_3 = \frac{2}{3}, l_2 = 4, l_3 = 2.$$

We let for simplicity $\nu_2 = +\infty$ that means that all switch-over times $\xi_2(n)$ are zeros. We further assume for simplicity that all $\sigma_1(n) = 0$ too.

We recall that u_4 is the fraction of time when the server is switching from queue 1 to queue 3, and that p is the fraction of the reduced cycles. In each of the following example, we run more than 10^8 cycles and find u_4 and p with the error smaller than $2 \cdot 10^{-4}$ with probability greater than 0.9999.

In what follows, a random variable with parameter, say C , “has a uniform distribution” means that it “has a uniform distribution on the interval $(0, 2/C)$ ”.

Example 1. *We assume that all interarrival, service, and switch-over times are uniformly distributed. We obtain $u_4 \approx 0.0466$ and $p \approx 0.1825$.*

Example 2. *We assume that the interarrival times to queue 2 have a probability density function $f(x) = 8/x^3$, $x \geq 2$. The other primitive random variables are assumed to be uniformly distributed. Then we obtain $u_4 \approx 0.0518$, $p \approx 0.2027$.*

Example 3. *We assume that the interarrival times to queue 2 have an exponential distribution with mean 4. The other primitive random variables are assumed to be uniformly distributed. Then we obtain $u_4 \approx 0.0619$ and $p \approx 0.2410$.*

Example 4. *We assume that the interarrival times to queue 2 have a probability density function $f(x) = ba^b/x^{b+1}$, $x \geq a$ with $a = 8 - 4\sqrt{2}$ and $b = 1 + \sqrt{2}$. The other primitive random variables are assumed to be uniformly distributed. Then we obtain $u_4 \approx 0.0446$ and $p \approx 0.1751$.*

a	p	u_4	a	p	u_4	a	p	u_4
0.18	0.4181	0.1097	0.50	0.3435	0.0892	1.5	0.2009	0.0514
0.19	0.4174	0.1095	0.55	0.3297	0.0855	2	0.1765	0.0450
0.20	0.4162	0.1091	0.6	0.3179	0.0825	2.5	0.1623	0.0413
0.25	0.4089	0.1071	0.7	0.2953	0.0763	3	0.1527	0.0388
0.30	0.3982	0.1041	0.8	0.2762	0.0712	4	0.1417	0.0360
0.35	0.3849	0.1005	0.9	0.2602	0.0670	5	0.1361	0.0346
0.40	0.3713	0.0969	1	0.2461	0.0632	10	0.1272	0.0322
0.45	0.3571	0.0929	1.25	0.2198	0.0563	20	0.1245	0.0315

Table 1: Simulation results for p and u_4 , varying the parameter a for the Weibull distribution.

We remark that the first two moments (4 and 32 respectively) of the interarrival times to queue 2 coincide in Examples 3 and 4. However u_4 and p are significantly different.

Example 5. *We assume that the interarrival times for queue 2 have a discrete distribution given by*

$$\mathbf{P}(\tau_2(1) = 4(2 - \sqrt{2})) = \frac{2 + \sqrt{2}}{4}, \quad \mathbf{P}(\tau_2(1) = 4(2 + \sqrt{2})) = \frac{2 - \sqrt{2}}{4}.$$

The other primitive random variables are assumed to be uniformly distributed. Then we obtain $u_4 \approx 0.0641$ and $p \approx 0.2494$.

We remark that the first three moments (4, 32 and 384 respectively) of the interarrival times to queue 2 coincide in Examples 3 and 5. However u_4 and p are different.

5.2 Examples with Weibull distribution

In this subsection, we again vary only the distribution of the interarrival times to queue 2. We focus on the class of Weibull distributions with a fixed mean. More precisely, we assume that the tail distribution function of the interarrival times for queue 2 is given by $\mathbf{P}(\tau_2(1) > x) = \exp(-bx^a)$, $x \geq 0$, with $b = (\Gamma(1 + a^{-1})/4)^a$. The other primitive random variables are assumed to be uniformly distributed. Note that $\mathbf{E}\tau_2(1) = 4$. Let $l_2 = 6$ and $l_3 = 4$.

Table 1 and Fig. 1 present the simulation results for p and u_4 , varying the parameter a for the Weibull distribution. The limiting value 0.1237 in Fig. 1 corresponds to the value of p when the interarrival times for queue 2 are 4, i.e., $\tau_2(n) = 4$, $n = 1, 2, \dots$, with probability 1. We observe that the lighter is the tail of the Weibull distribution, the smaller are the values p and u_4 .

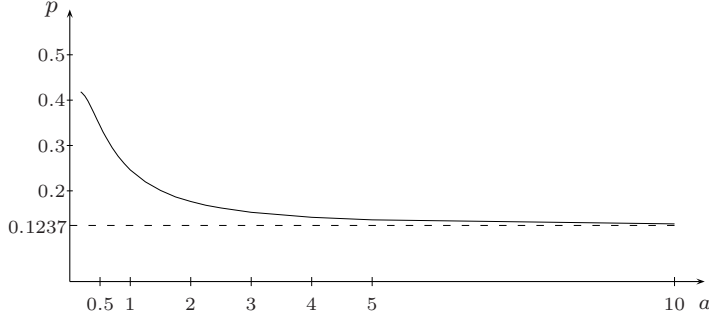


Figure 1: Simulation results for p , varying the parameter a for the Weibull distribution.

6 The fluid model and proof of Theorems 1 and 2 for gated and exhaustive service disciplines

In this section, we consider the polling system with either gated or exhaustive service disciplines. The fluid limits and the fluid limits with fixed initial states are defined as in Section 3. The following lemma can be shown by the standard procedure.

Lemma 10. *For every fluid limit $\bar{\mathbf{X}}(t) = (\bar{Q}(t), \bar{T}(t), \bar{U}(t))$, the following equations are satisfied:*

$$\bar{Q}_k(t) = \bar{Q}_k(0) + \lambda_k t - \mu_k \bar{T}_k(t), \quad k = 1, 2, 3, \quad t \geq 0; \quad (6.1)$$

$$\bar{Q}_k(t) \geq 0, \quad k = 1, 2, 3, \quad t \geq 0; \quad (6.2)$$

$$\bar{T}_k(\cdot) \text{ and } \bar{U}_j(\cdot) \text{ are nondecreasing, } \quad k = 1, 2, 3, \quad j = 1, 2, 3, 4; \quad (6.3)$$

$$\sum_{k=1}^3 \bar{T}_k(t) + \sum_{j=1}^4 \bar{U}_j(t) = t, \quad t \geq 0; \quad (6.4)$$

$$\text{If } |\bar{Q}(t)| > 0, \text{ then } \sum_{k=1}^3 \bar{T}'_k(t) = 1, \quad t \geq 0. \quad (6.5)$$

For the polling system with either gated or exhaustive service disciplines, we call the equations (6.1)-(6.5) the *fluid model equations* and call a solution $\bar{\mathbf{X}} = \{(\bar{Q}(t), \bar{T}(t), \bar{U}(t)) : t \geq 0\}$, of the fluid model equations a *fluid model solution*.

Using the similar argument as in the proof of Lemma 4, we can prove the following result.

Lemma 11. *Suppose that the Markov process X is positive Harris recurrent, and a fluid limit with fixed initial state $\bar{\mathbf{X}}(t) = (\bar{Q}(t), \bar{T}(t), \bar{U}(t))$ is driven from a sample path ω that satisfies (3.22). Then we have*

$$\bar{U}'_k(t) > 0, \quad t \geq 0, \quad k = 1, 2, 3, 4. \quad (6.6)$$

We call the equations (6.1)-(6.5) plus (6.6) the *augmented fluid model equations* and call a solution $\bar{\mathbf{X}}$, to these equations an *augmented fluid model solution*.

Definitions 1 and 2, and Lemmas 2, 3 and 5 can be applied to the polling systems with gated and exhaustive service disciplines. Therefore Theorems 1 and 2 for gated and exhaustive service disciplines are proved by Propositions 4, 5 and 6 below.

Proposition 4. *For the polling system with gated or exhaustive service disciplines, the fluid model is stable if $\rho_0 < 1$.*

Proof. For a fluid model solution \bar{X} , let $W(t) \equiv \sum_{k=1}^3 \frac{\bar{Q}_k(t)}{\mu_k}$. By (6.1) and (6.5), $W'(t) = \rho_0 - 1$ if $W(t) > 0$. Hence the fluid model is stable if $\rho_0 < 1$. \square

Proposition 5. *For the polling system with gated or exhaustive service disciplines, the augmented fluid model is weakly unstable if $\rho_0 \geq 1$.*

Proof. For an augmented fluid model solution \bar{X} , let $W(t) \equiv \sum_{k=1}^3 \frac{\bar{Q}_k(t)}{\mu_k}$. By (6.1), (6.4) and (6.6), $W'(t) > \rho_0 - 1$. Hence the augmented fluid model is weakly unstable if $\rho_0 \geq 1$. \square

Proposition 6. *For the polling system with gated or exhaustive service disciplines, the fluid model is weakly unstable if $\rho > 1$.*

Proof. For a fluid model solution \bar{X} , let $W(t) \equiv \sum_{k=1}^3 \frac{\bar{Q}_k(t)}{\mu_k}$. By (6.1), (6.3) and (6.4), $W'(t) \geq \rho_0 - 1$. Hence the fluid model is weakly unstable if $\rho > 1$. \square

The research of N. Chernova was partially supported by the Ministry of Higher Education and Science of the Russian Federation Grant RNP.2.1.1.346. The research of S. Foss was partially supported by the London Mathematical Society travel grant and by the Royal Society International Joint Project. The research of B. Kim was supported by the Korea Research Foundation (KRF) grant funded by the Korea government (MEST) (2009-0076600).

References

- [1] S. C. Borst (1995). Polling systems. CWI tracts, 115. Centrum voor Wiskunde en Informatica, Amsterdam.
- [2] O. J. Boxma, J. Bruin, B. H. Fralix (2009). Sojourn times in polling systems with various service disciplines. Perform. Eval., 66, 621–639.
- [3] M. Bramson. Stability of two families of queueing networks and a discussion of fluid limits. Queueing Systems, 28, 7–31.
- [4] H. Chen, A. Mandelbaum (1991). Discrete flow networks: Bottlenecks analysis and fluid approximations. Math. Oper. Res., 16, 408–446.
- [5] J. G. Dai (1995). On positive Harris recurrence of multiclass queueing networks: A unified approach via fluid models. Ann. Appl. Probab., 5, 49–77.
- [6] J. G. Dai (1996). A fluid limit model criterion for instability of multiclass queueing networks. Ann. Appl. Probab., 6, 751–757.

- [7] J. G. Dai, J. J. Hasenbein, B. Kim (2007). Stability of Join-the-Shortest-Queue Networks. *Queueing Systems*, 57(4), 129–145.
- [8] J. G. Dai, S. P. Meyn (1995). Stability and convergence of moments for multiclass queueing networks via fluid limit models. *IEEE Transaction on Automatic Control*, 40, 1889–1904.
- [9] D. G. Down (1996). On the stability of polling models with multiple servers. *J. Appl. Probab.*, 35, 925–935.
- [10] S. Foss, A. Kovalevskii (1999). A stability criterion via fluid limits and its application to a polling system. *Queueing systems*, 32, 131–168.
- [11] A. N. Rybko and A. L. Stolyar (1992). Ergodicity of stochastic processes describing the operation of open queueing networks. *Problems of Information Transmission*, 28, 3–26.
- [12] A. L. Stolyar (1995). On the stability of multiclass queueing networks: A relaxed sufficient condition via limiting fluid processes. *Markov Processes and Related Fields*, 1, 491–512.
- [13] V. M. Vishnevsky, O. V. Semenova (2008). Adaptive dynamical polling in wireless networks. *Cybernetics and Information Technologies*, 8, 3–11.
- [14] A. Wierman, E. M. M. Winands, O. J. Boxma (2007). Scheduling in polling systems. *Perform. Eval.*, 64, 1009–1028.
- [15] E. M. M. Winands, I. J. B. F. Adan, G. J. van Houtum, D. G. Down (2009). A state-dependent polling model with k -limited service. *Prob. Eng. Inform. Sci.*, 23, 385–408.